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# Abstract

Document JCTVC-R0139 proposes a new SEI message aiming at using a modulation channel in combination with a limited bit-depth channel. The current document completes JCTVC-R0139 and presents in details the decomposition side of the proposed modulation process. The usage of a modulation channel allows for coding high bit-depth or even floating point video sequences with low bit depth encoding/decoding devices. The concept consists in splitting an input high bit-depth signal into two signals of low bit-depth, one modulation signal and one low bit-depth (LBD) signal corresponding to the residual part of the modulation. In the proposed approach, two functional modes are considered, one guaranteeing the LBD signal to be conform to BT.709 or BT.2020 recommendations (therefore being usable as is on BT.709 or BT.2020 rendering devices), and one offering optimal coding efficiency but without direct compatibility with BT.709 or BT.2020 rendering devices.

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# Introduction

This proposal addresses the topic of high bit-depth (HBD), or even floating point (FP) video signal coding, using a modulation approach. The document completes document JCTVC-R0139, which describes a new Modulation channel information SEI message that enables the implementation of this solution. In this current document, we focus on the description of the pre-processing step to be applied at the encoding side for demodulating the input HBD signal. The process applies per picture. The demodulating consists in splitting the input picture PHBD into two separate and independent layers:

* a low bit depth monochromatic modulation picture Pmod
* a standard low bit depth three color picture PLBD, typically coded in 10 bits.

The original signal is the multiplication of these two layers, such one gets per component:

PHBD = Pmod\* f-1(PLBD)

where f is a non-linear mapping used to derive the LBD layer from the multiplicative residual PHBD / Pmod.

The encoder pre-processing is implemented in two basic modes, YCbCr mode where the LBD picture is in YCbCr format and can be rendered on devices conform to BT.709 or BT.2020 recommendations, and Lab mode, offering enhanced compression capabilities, but where the LBD picture cannot be correctly rendered without additional mapping.

The decomposition process is schematically depicted in the Figure 1. The input HBD video is considered to be linear (if not, a pre-conversion applies to convert the signal in the linear domain). From the input HDB video, the modulation picture is generated. The corresponding process is described in section 2. From the input HDB video and the modulation picture, the LBD picture is then generated. This process is described in section 3. In both cases, the process is described for both YCbCr and Lab modes.



Figure 1: simplified block synoptic of the encoding process.

The process for generating the modulation picture is the same for YCbCr and Lab modes (with different parameters settings). The idea is to create a modulation picture representative of the low frequency part of the luminance component of the HBD video. From there, a multiplicative residual is computed by dividing the HBD picture by the associated modulation picture and, by using suitable color transforms and mappings, a LBD picture is created. The outputs of the decomposing process are:

* an LBD picture
* a modulation picture
* some parameters to control the recomposing process.

# Modulation picture generation process

## Modelling of the modulation picture by separable Shape functions

The modulation picture is generated from the linear luminance component Y of the HBD video. In case the input format is linear XYZ, this component is simply the input Y. Otherwise, a pre-processing is needed to get the linear Y component, for instance from a gammatized RGB input.

The process consists in extracting a smooth (low frequency) version of the logarithm luminance picture L = log(Y). Several solutions may be used for this smoothing process. The option proposed in this contribution is to model the modulation picture as a linear combination of pre-defined Shape Functions (SFs), as illustrated in Figure 2. The SFs are mathematically built as partition of unity such that the luminance picture is fitted at best with a low amount of data and enough smoothness. Prior to this modeling process, a thresholding is first applied in order to avoid extreme values from influencing the SF-model fitting.

Let ψi be a SF (one-dimensional coefficients array) with weight ai. The modulation picture Pmod is the linear combination of coefficients ai by SF ψi , for i corresponding to a sparse sampling of the picture grid. The coefficients ai are identified by least mean square method as described in the next sub-section.

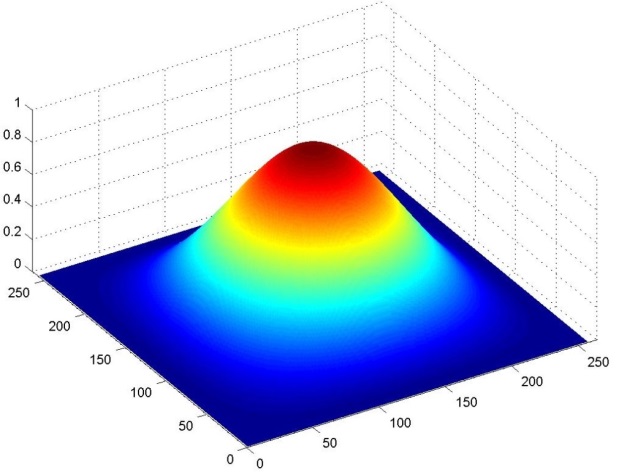
 

Figure 2: left: modulation picture modelling by SFs – right: used SF shape.

## Determination of the weights ai of the SF

For notation convenience, we represent each SF ψi by a column vector whose length is the number of pixels of the image. Similarly, the luminance map Lfit is represented by a column vector . Let N be the number of SF’s. The array of weights *a* is identified by least mean squares as:

where

which leads to the well-known solution:

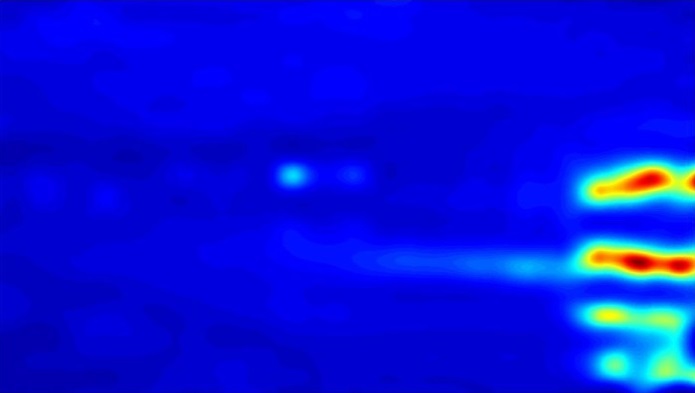
Note that the matrix *M* is sparse because the support of the SF’s is small. This makes the product fast to compute.

### Spatial stabilization of the modulation picture

If the modulation picture is not spatially stabilized, one gets rebound effect around strong gradients. This possibly leads to annoying spatial artifacts in the LBD picture, like halo effects around objects with strong luminance gradients at their boundaries. Also, the multiplicative residual suffers from these rebounds and may be harder to predict both spatially and temporally, leading to a less compressible residual. To control this phenomena, a spatial stabilization of the modulation picture is used. The method consists in adding a stabilization term in the minimization problem as follows:

where is a Laplace operator applied on the weights a’. The parameter μ tunes the strength of the stabilization (bigger μ leads to a flatter modulation picture as shown in Figure 3). This minimization problem can be solved as:

where *D* is a NxN matrix which is a discrete version of the Laplace operator Δ.

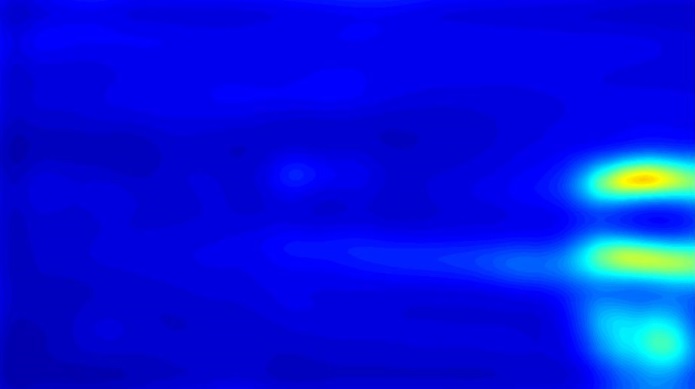
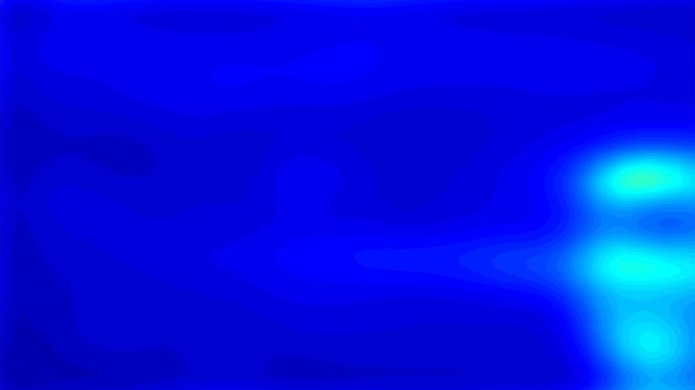
 

Figure 3:  source picture and its corresponding modulation picture for increasing μ values.

### Temporal stabilization of the modulation picture

An additional temporal stabilization may be required to control fast brightness changes. The main idea is to add a stabilization term in the minimization problem as follows :

where is the coefficient vector of the modulation picture of a previous reference frame. This minimization problem can be solved as:

## Final reconstruction of the modulation picture

The modulation picture in the linear domain is reconstructed from the determined weights ai’s by using the formula:

The exponential function is used because the LMS fit has been performed in the log domain and not the linear domain Y.

There are mainly two strategies to transmit the modulation picture to the decoder:

1. either send the coefficients ai’s and let the decoder reconstruct the modulation picture; this first option is not considered in the recomposing process described in JCTVC-R0139;
2. or send directly a compressed version of the reconstructed modulation picture, for instance using the frame packing or auxiliary picture mechanisms (cf JCTVC-R0139 for more details); the modulation picture Pmod is quantized in the log domain (before applying the exp operator), after a possible clipping to avoid too small and large values.

## Parameters of the modulation picture generation process

Typical values for the SF topology are a spacing of several 10s of pixels, with an overlap of 1 to 3 (1 SF overlaps its neighboring SFs by 1 or 3 SF width/height). As an example in Figure 2, the SF overlap is of 1 horizontally and 1 vertically. The other parameters mostly depend on mode. In YCbCr mode², the viewability of the LBD picture is crucial, and strong spatial stabilization and temporal stabilization are used. Typical values are  around 3000,  around 1500. For the Lab mode, compression capability is preferred to the viewability and the regularization constraints are reduced. Typical values are  around 0,  around 500. A final clipping applies before quantization of the modulation picture in order to avoid dark/bright unbalance in the multiplicative residual. Typical value is around 100 in the YCbCr case, and 10000 in the Lab case.

# LBD picture generation process

For the LBD picture generation, a multiplicative residual is first computed from the linear XYZ input and the reconstructed modulation picture Pmod by using a division of the linear input by the modulation picture. Then, a mapping and a color space transform are used to obtain a LBD picture. This process is described for the YCbCr and Lab modes in the following sub-sections.

## GLog mapping

Despite the range reduction due to the division by the modulation picture, very bright pixels may still be present in the residual picture because of specular light or very bright small objects like fireworks or lights on a dark background. It is known that a gamma correction does not flatten high lights fast enough to avoid burning of bright pixels after clipping; that is why Log-based correction (noted GLog) is used instead:

GLog( x ) = a \* ln( x + b ) + c.

The parameters a, b, c of the GLog curves are determined such that GLog(0) = 0, GLog(1) = 1, and the derivative in 1 is the same as a gamma function: GLog’(1) = γ. Note the γ parameter here is only used to configure the above GLog mapping function in a practical way. Hence the proposed mapping process has nothing to do with a traditional gamma correction. Practical examples are shown in Table 1 and in Figure 4. One observes that high lights are lowered much more aggressively with a GLog curve than with a gamma curve.

Table 1: different settings for the GLog transfer function.

|  |  |  |  |
| --- | --- | --- | --- |
| **γ** | **a** | **b** | **c** |
| 1/2.0 | 0.6275 | 0.2550 | 0.8575 |
| 1/2.4 | 0.4742 | 0.1382 | 0.9386 |
| 1/2.8 | 0.3861 | 0.0811 | 0.9699 |

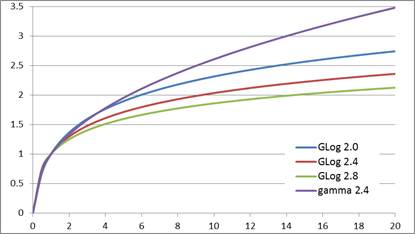


Figure 4: GLog curves

## LBD picture generation process in YCbCr mode

In YCbCr mode the generated LBD picture is under the standard color space YCbCr. The method presented here is not limited to the BT.709 chromaticities and may be straightforwardly adapted to other chromaticities like BT.2020 for instance.

### Overview

Here is a sketch of the main steps of the LBD layer generation and coding. The following processes are performed as shown on Figure 5.

1. input of the linear 4:4:4 XYZ HBD video (if necessary after conversion)
2. intensity modulation of the luminance picture Lfit into Lfit,mod
3. generation of the modulation picture Pmod as described above
4. transformation, in the linear domain, of the XYZ HBD pictures into RBG pictures; depending on the chromaticities (e.g. BT.709, BT.2020)
5. division, in the linear domain, of the three components RGB by the modulation picture Pmod to get residual components

Rres= R / Pmod, Gres= G / Pmod, Bres= B / Pmod,

1. tone mapping of Rres, Gres, Bres onto N (=8, 10 or 12) bits by using the parametric tone mapping described in the next section

R’res= TM( Rres ), G’res= TM( Gres ), B’res= TM( Bres ),

1. transformation of the tone mapped R’res G’res B’res components into YCbCr components by using a standard LBD transform RGB🡪YCbCr
2. the resulting YCbCr picture is the LBD picture to be sent to a standard N bit encoder like h264/AVC or HEVC; at this stage chroma down-sampling 4:4:4🡪4:2:2 or 4:4:4🡪 4:2:0 may be performed

The idea behind this scheme is to mimic the LBD scheme where RGB linear components are gammatized to R’G’B’ before being transformed into YCbCr. We show in next section how to tune the parametric tone mapping to obtain a viewable LBD YCbCr layer.

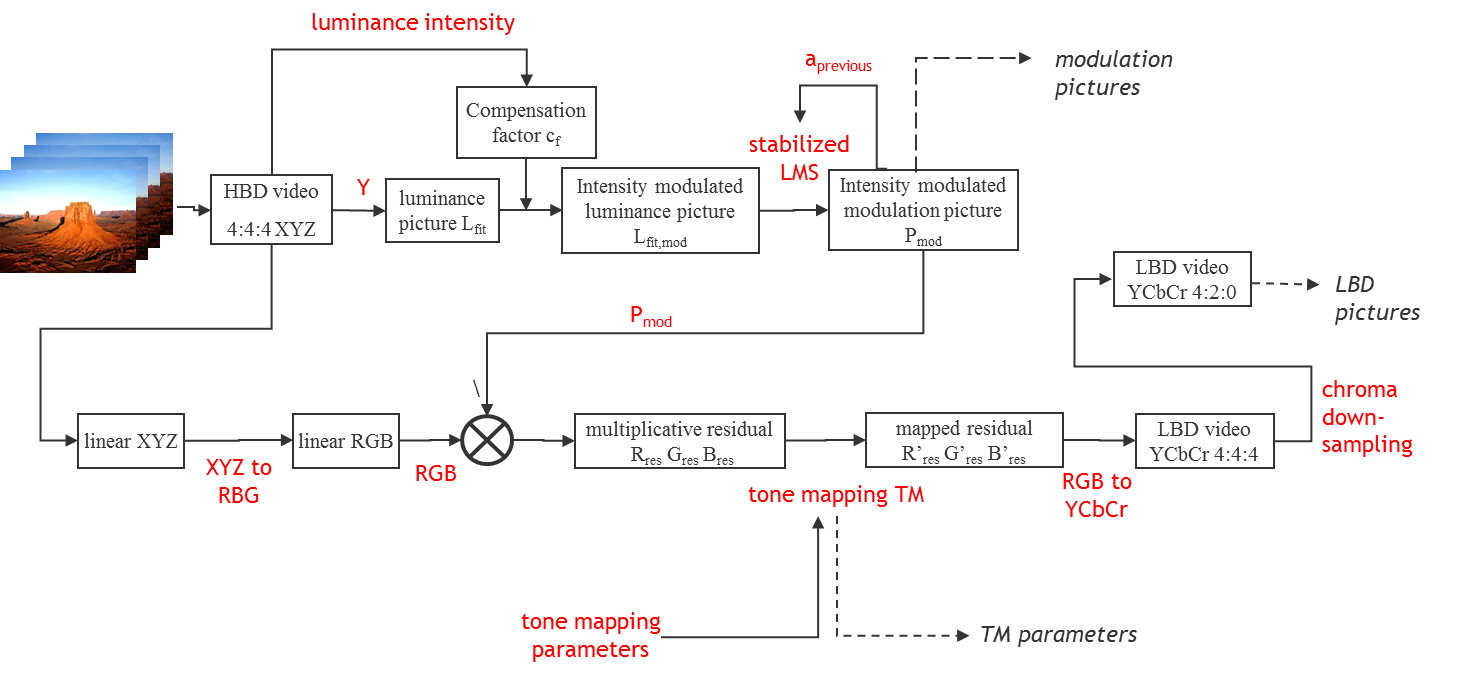


Figure 5: decomposition process in mode YCbCr

### Intensity modulation of the modulation picture

By construction of the modulation picture, the residual RresGresBres has a mean value close to one, and this independently from the image content. As a consequence, all images, dark or bright, will lead to a residual with roughly the same mid brightness. Obviously, this breaks the consistency with the original video. To solve this issue, the LBD signal is modulated such that bright scenes look bright, and dark scenes look dark on the LBD video sequence. The intensity modulation is directly applied to the luminance map Lfit before determination of the weights ai’s by the stabilized LMS process. This consists in the following operation:

Lfit,mod = cf Lfit/E(Lfit)

where the factor cf is adjusted based on the mean value of the luminance E(Lfit). Typically cf is proportional to  (E(Lfit))1/2 .

### Parametric tone mapping

The tone mapping performs the following operation before clipping the tone mapped data to N bits:

R’res= TM( Rres ) = GLog( Rres \* (LumaBalanceFactor/10)) \* ScalingFactor

the GLog transfer function being applied with γ=1/*GLog* (identical formulas apply for the green and blue components). The parameters are derived as follows:

* The final scaling by *ScalingFactor* is set such that the LBD tone mapped data occupies the full N bit range and is tuned to obtain a global luminosity consistent with the scene. Typical value is around 50-60.
* The parameter *GLog* is chosen to be close to a standard gamma curve in the mid lights such that the hue is preserved and no shift in colors is noticeable between the HDR video and the tone mapped viewable LBD video. Typical value is around 2.5.
* The pre-scaling by *LumaBalanceFactor* is used to choose the balance between the dark and the bright regions of the picture. A high value tends to give more room to the dark regions and to burn the bright regions. On the other hand, a low value tends to provide more details in bright regions but very dark regions are pushed down close to zero and lack details. Typical value is around 40-60.

## LBD picture generation process in Lab mode

In the Lab mode, compression performance is of highest priority and the viewability of the LBD layer is not considered. A dedicated quantization space, namely the Lãᵬ, is used to maximize the compression performance. The GLog transfer function parameters are fixed in order to obtain a LBD layer with the most details allowing to reach very high quality compressed videos. Finally, local adaptation of the GLog mapping, driven by a local theoretical input maximum, allows to further enhance compression capabilities.

The construction of the Lãᵬ color space is explained in details in the Annex.

### LBD picture generation using the Lãᵬ space

Here is a sketch of the main steps of the LBD layer generation and coding. The following processes are performed as shown on Figure 6.

1. input of the linear 4:4:4 XYZ HBD video
2. generation of the modulation picture Pmod as described above
3. division, in the linear domain, of the three components XYZ by the modulation picture Pmod to get residual components in the D65 illuminant

Xres= X / ( 0.9505 \* Pmod ), Yres= Y / Pmod, Zres= Z / ( 1.0890 \* Pmod ),

1. tone mapping of Rres, Gres, Bres onto N (=8, 10 or 12) bits by using a fixed GLog mapping f

X’res= f( Xres ), Y’res= f( Yres ), Z’res= f( Zres ),

1. transformation to a Lab-like space

L = 116 \* Y’res, a = 500 \* ( X’res -Y’res ), b = 500 \* ( Y’res - Z’res )

1. transformation to the associated Lãᵬ

C² = a² + b², Ĉ = ln( 1 + k \* C ) / k, ã = a \* Ĉ / C and ᵬ = b \* Ĉ / C

1. scaling by a multiplicative scaling factor s

Lout = s \* L, ãout=s \* ã and ᵬout=s \* ᵬ

1. the resulting Lãᵬ picture is the LBD picture to be sent to a standard N bit encoder like H.264/AVC or H.265/HEVC; at this stage chroma down-sampling 4:4:4🡪4:2:2 or 4:4:4🡪4:2:0 may be performed.

The parameters are typically set to: a0 = 0.4481, b0 = 0.1203, c0 = 0.9491, k=0.03, scal0 = 0.4408, and s = 256 / 116 \* 2N-8 \* scal0, N being the LBD signal bit-depth.

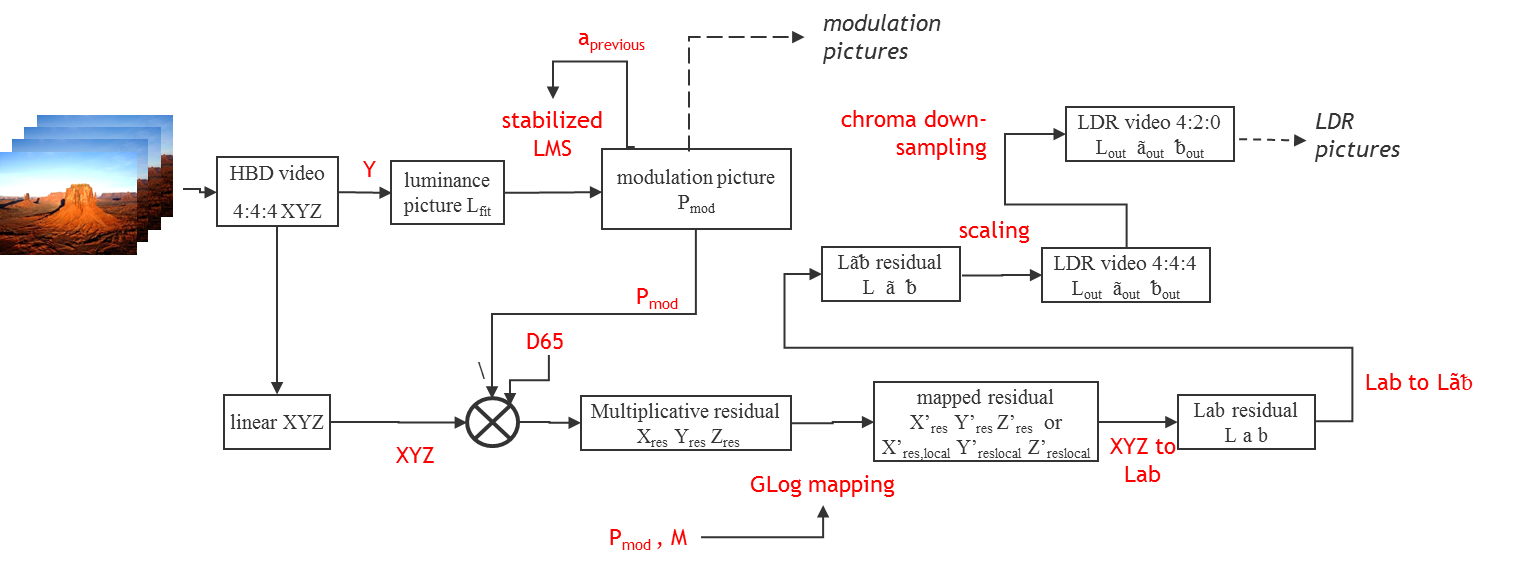


Figure 6: decomposition process in mode Lab

### Local maximum adaptation

In order to improve the representation of the LBD picture by occupying as best as possible all possible code-words, the mapping f is localized and made dependent on the modulation picture. Let M be the upper bound for the value Y. Knowing that Y’res ≤ f(M/Pmod), the output signal Lout is actually bounded by ( 2N \* scal0 \* f( M / Pmod ) ). The signal is therefore normalized using the following factor

norm = Clip[0.5,2]( scal0 \* f( M / Pmod ) )

the clipping in [0.5, 2] being introduced in order to avoid too strong local correction leading to too much details in the dark.

Practically, the value for M is taken from the input format upper bound (M=10000 nits in the current implementation of the decomposing process). Figure 7 illustrates the coding efficiency gain from this renormalizing.

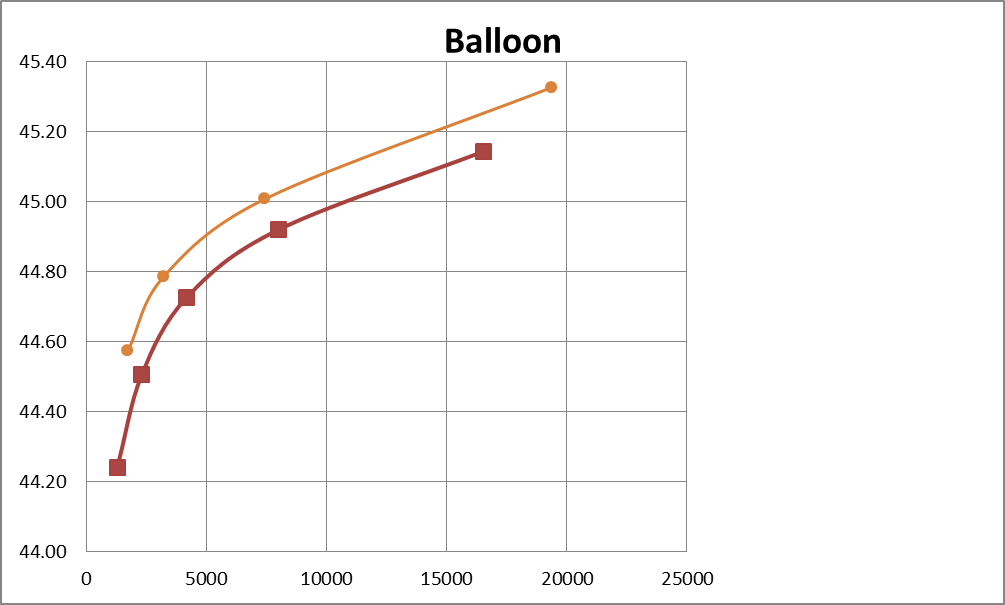


Figure 7: effect of the local mapping fPmod (orange) compared to the global mapping f (red) on the test sequence Balloon. Curves are kb/s vs. PSNR Lab2000 (dB)

# Patent rights declaration(s)

**Technicolor may have current or pending patent rights relating to the technology described in this contribution and, conditioned on reciprocity, is prepared to grant licenses under reasonable and non-discriminatory terms as necessary for implementation of the resulting ITU-T Recommendation | ISO/IEC International Standard (per box 2 of the ITU-T/ITU-R/ISO/IEC patent statement and licensing declaration form).**

# Annex: the Lãᵬ color space

A color space is said to be perceptual if

* it is three dimensional and has for components
  + one luminance L
  + two chrominances C1 and C2
* a distance d is defined on the space, i.e. this is a metric space
* the distance between two points (L,C1,C2) et (L’,C’1,C’2) is proportional to the visual perceptual difference between these two points

Of course, this an imperfect mathematical experimental construction which is supposed to vary depending on each individual. However, we will suppose that the perceptual space is perfect; in particular there exists a maximum perceptual bound ΔE below which human beings cannot perceive any difference two colors if the following inequality is

d((L,C1,C2), (L’,C’1,C’2))< ΔE.

The perceptual color space is said to be Euclidian if the metric is Euclidian. This means that the metric follows the formula

d((L,C1,C2), (L’,C’1,C’2))² = (L-L’)² +( C1- C’1)² +( C2- C’2)²

A well-known example of Euclidian perceptual space is the Lab76 described hereafter. It is deduced from the universal space XYZ by using the following transform for a given triplet (Xn, Yn, Zn) of reference lightning. The transform to the perceptual space Lab CIE1976 is performed by the equations

* L = 116f(Y/Yn)-16
* a = 500(f(X/Xn)-f(Y/Yn))
* b = 200(f(Y/Yn) – f(Z/Zn))

where f is essentially a gamma function of order 1/3, say

f(r) = r^(1/3) if r>(6/29)^3,

f(r) = 1/3\*(29/6)^2\*r + 4/29 otherwise.

It has been established that, in this space, two colors are humanly indistinguishable, under the lightning conditions (Xn, Yn, Zn), as soon

d76((L,a,b),(L’,a’,b’))² := (∆L)² + (∆a)² + (∆b)² < (∆E0)²

where ∆L is the L-distance for these two colors, ∆a is the a-distance and ∆bis the b-distance. A typical value for ∆E0 is between 1 and 2.

The distance is Euclidian, this means that a classical codec can work directly in this space without sacrificing the trade-off bitrate vs. perceptual quality. However, the Lab76 space has turned out to be not really perceptually uniform as the metric is not proportional to the perceived visual difference. Thus, this space has been improved into the Lab94 but without changing the Lab definition: only the metric d has evolved. This new metric is given by

d94((L,a,b),(L’,a’,b’))² := (∆L)² + (∆C/(1+k1C))² + (∆H/(1+k2C))²

where C is the saturation and h is the hue. These new quantities are provided by the relations

C² :=a²+b², h = atan(b/a) et H = Ch.

Clearly, the distance is not Euclidian anymore as soon as k1 and k2 are non-zero. From here, we will show that a naïve quantization of the (L,C,h) is not compatible with the perceptual distance. Actually, we will simply prove that for the component C taken alone.

Let us try

Ĉ := C/(1+k1C) <-> C= Ĉ/(1-k1Ĉ)

and encode “naively” Ĉ with a quantization step ΔĈ. One gets, from a first approximation,

C ≈ (Ĉ+ΔĈ)/(1-k1(Ĉ+ ΔĈ)) ≈ C(1+ΔĈ/Ĉ)/(1- k1Ĉ/(1-k1Ĉ)) ≈ C(1+ ΔĈ(1/Ĉ+ k1Ĉ/(1-k1Ĉ)))

and identification of the first order terms leads to

ΔC = CΔĈ(1/Ĉ+ k1Ĉ/(1-k1Ĉ)) = CΔĈ(1+k1C)/Ĉ = ΔĈ(1+k1C)²

As a consequence, if Ĉ is quantized up to ΔE, then the error on C is up to ΔE(1+k1C)² et not the desired ΔE(1+k1C) as one would expect if being compatible with the perceptual distance. This is due to the fact that the inverse transform

C= Ĉ/(1-k1Ĉ)

is **not** linear. We will show that it is possible to find a quantization space for C such that the error on this component is the desired ΔE(1+k1C). Let us take the problem the other way and let us look for a transform f such that

Ĉ = f(C) et ΔC=ΔĈ(1+k1C).

Assuming some regularity on f, a first order approximation provides

C = f-1(Ĉ) ≈ f-1 (Ĉ+ΔĈ) ≈ f-1(Ĉ)+ ΔĈ f-1’(Ĉ) = C+ΔĈ f-1’(Ĉ)

and by a first order identification, it follows

ΔĈ(1+k1C)=ΔC = ΔĈ f-1’(Ĉ) => 1+k1C=f-1’(Ĉ)=1/f’(C) => f’(C)=1/(1+k1C)

and then, an integration gives

Ĉ = f(C)=ln(1+k1C)/ k1.

This is the good quantization space for C taken alone if one uses the perceptual metric of the Lab94 space.

Unfortunately, in the case of several components, solving the associated differential form is not straightforward. From here, we show that despite the lack of analytic solution it is possible to find out some closed formulation but slightly sub-optimal for the quantization space of the two components a and b in the Lab94 space.

First, let us have a look at the two constants k1 and k2 in the distance

d94((L,a,b),(L’,a’,b’))² := (∆L)² + (∆C/(1+k1C))² + (∆H/(1+k2C))².

Their recommended values are about k1=0.045 and k2=0.015. Let us define k :=min{k1,k2}. One defines a more constraining distance by

d’94((L,a,b),(L’,a’,b’))² := (∆L)² + (∆C/(1+kC))² + (∆H/(1+kC))² ≤ d94²

such if one gets a given level of quality in the new Lab94’ space, this level of quality is at least reached in the Lab94 space. Now, Pythagoras’ theorem gives

d’94²= (∆L)^2 + (∆a/(1+kC))² + (∆b/(1+kC))².

One would like to use a uniform 2D quantization step ∆E times ∆E such that the respective errors on a and b are compatible with (1+kC) ∆E. Let us name ã et ᵬ the two associated components of the quantization space, and let us define ϕ the transform associated to this quantization space

ϕ(a,b) = (ã,ᵬ)

We suppose that this transform is smooth enough to allow the following calculation and first order approximations



and by first order identification, one gets



where g(C):=1+kC. After a slight simplification , one deduces the following equality involving the Jacobian matrix J and the perceptual weight 1+kC

. (\*)

Let us recall the definition of the Jacobian matrix in term of partial derivatives

.

Also, by symmetry on a and b, one has necessarily ϕa(b,a)= ϕb(a,b). Thus, from the determination of ϕa, one will get ϕb automatically by symmetry. The first row of (\*) provide a Partial Differential Equation (PDE) on ϕa

 (\*\*).

Unfortunately, there is no simple integral formulation for the solutions of (\*\*). So, we try the Ansatz

ϕa(a,b) = aψ(C)

such that, using  et , equation (\*\*) becomes

.

One approximates the term (a²+ab)/C by C; the validity of the solution found using the Anstaz and the approximation will be proved a posteriori. One gets



which is a linear differential equation on ψ. A particular solution is ψ0=cst/C. Now, using the well-known constant variation technique, one proves that the solutions are

ψ(C) = ln(1+kC)/kC = Ĉ/C.

Finally, the quantization space is

ã=aĈ/C and ᵬ=bĈ/C

This is the quantization space associated to (a,b) in the Lab94 space. One checks, after lengthy calculation and inequalities, that



and it becomes clear that the quantization space is sub-optimal with at most a factor square root of two, which is good on the paper.